The role of inflation in retirement planning – why reducing nominal risk can increase real risk

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Abstract

In order to make good decisions in retirement planning, consumers' and advisors' need to understand the return potential and the risks of the respective products. Usually, risk-return indicators are based on nominal measures, derived from the probability distribution of nominal wealth at the end of the product's term. For consumers, however, real benefits (i.e., the benefits in 'today's purchasing power') are more relevant than nominal benefits. We show that real risk-return characteristics can be structurally different from nominal risk-return characteristics, in particular for products that come with some nominal guarantees.

Firstly, we derive from economic arguments and existing literature why the return of certain assets, particularly stocks, exhibits a positive correlation with inflation over long periods of time. We argue that such long-term effects need to be considered in an analysis of long-term savings processes. Secondly, we introduce a capital market model where expected equity returns depend on inflation implying such a positive correlation over long time horizons and analyze how fundamental (simple) results change if we focus on real returns. For instance, we show that in our model the equity ratio of the utility maximizing portfolio in the famous Merton-problem does not change when utility of the real benefit is considered whereas the real risk and return of a simple mix between a bond and an equity investment can heavily deviate from the nominal result (particularly for long time horizons). We also analyze how the real risk-return characteristics of typical retirement savings products structurally deviate from their nominal counterparts. We particularly show that, under certain circumstances, an increase in (nominal) guarantees can increase real risk. Note, we assume that – as is currently the case in practice – guarantees of typical products are given in nominal terms (and bonds with nominal notional values are used as 'safe assets'). Finally, we derive implications for consumers, financial advisors, and policy makers.

Keywords: Risk Return, Financial Planning, Retirement Planning, Inflation, Investment Products, Guarantee Products.

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1 Introduction

Finding the optimal asset allocation for long term savings processes, e.g., in the context of retirement planning, is very complex and has been intensively discussed in the literature.[1] However, theoretically optimal solutions are in general not practically feasible (or cannot even be calculated since consumers are unable to specify their utility function or bequest motive). Hence, in practice consumers (and/or their financial advisors) are not so much interested in finding **the optimal asset allocation** but rather **a suitable product** out of a range of available products. This argument is in particular used in Graf et al. (2012) as a motivation to use so-called risk return-profiles (i.e., a suitable visualization of the probability distribution of a product's benefits) when explaining products to consumers.

In order to improve consumers' and advisors' ability to make decisions in retirement planning, they need to gain a sound understanding of the return potential and the associated risks of the respective products. Consequently, more and more transparency regulation has become effective in many countries (e.g., the Pan-European requirement of drawing up key information documents for so-called PRIIPs[2]). Typically, under such regulation, certain allegedly simple, meaningful and easy to understand risk-return indicators have to be disclosed to the consumer before a product is purchased.

Usually, risk-return indicators are based on **nominal** measures, e.g., derived from the probability distribution of **nominal** wealth at the end of the product's term.[3] Such nominal measures can educate consumers that a product with a higher expected return is in general also riskier. For instance, if in a fund with a constant mix between equity/stock investments and bonds (denoted as 'balanced fund' in what follows) the equity ratio is increased, both, the product's expected return and the product's risk typically[4] increase. Similarly, if in some guaranteed product the guarantee level is increased, then typically, both, the expected return

[1] Cf. e.g. Cairns et al. (2006) or Gerrard et al. (2010) and references therein to name only very few who solve the asset allocation problem in the context of retirement planning within an expected utility setting.

[2] Cf. European Union (2014).

[3] The requirements for the so-called Pan-European Personal Pension Product (PEPP) are an exception, since here inflation risk shall be assessed as part of the product disclosure. However, a model used in a corresponding paper by EIOPA (cf. EIOPA, 2020) does not consider any correlation between inflation and investment returns. Hence, in this model inflation impacts the risk-return characteristic of all products in the same way.

[4] There is, of course, an exception if the equity ratio is close to 0, where an increase of this ratio can lead to a decrease in risk due to diversification effects.

and the risk decrease. The message to the consumer seems to be crystal-clear: 'A reduction in the expected return is the price you pay for a higher degree of safety.'

In the present paper, however, we argue that for a typical consumer, **real benefits** (i.e., the benefits in 'today's purchasing power') are more relevant than nominal benefits. We further show that real (i.e., inflation-adjusted) risk-return characteristics can be structurally different from nominal risk-return characteristics. In particular, we show that the transition from nominal to real risk-return characteristics can impact different products in a different way. Under certain circumstances, a product that is less risky than a competing product in nominal terms can be riskier than the competing product in real terms. Under such circumstances, a risk averse consumer who is willing to give up some return potential in order to 'buy' additional safety, would actually give up return potential and end up with even more risk. Hence, typically used nominal risk-return indicators can potentially misguide consumers – particularly in long-term savings processes which are relevant in the context of old-age provision.

We would like to stress that our focus is not on the (trivial) statement that the real value of future wealth of any investment will be diminished by inflation. From this trivial statement it is often deduced, that the 'target wealth' at the retirement date should be adjusted for **expected inflation**. Our focus, in contrast, is on the **uncertainty of inflation** and its potentially different impact on real risk-return characteristics of different investments: A product that uses assets that are correlated with inflation tends to have a higher (lower) payout if inflation is high (low) during the term of the product. Such a product is less risky in real terms than a product that comes with the same probability distribution of terminal wealth in nominal terms but has a lower correlation with inflation.

In this paper, we analyze different aspects of nominal vs. real risk-return characteristics that are relevant for long-term savings processes in retirement planning. Firstly, we derive from economic arguments and existing literature why the return of certain assets exhibits a positive correlation with inflation **over long periods of time**. Secondly, we introduce a capital market model where expected equity returns depend on inflation implying such a positive correlation over long time horizons and analyze how fundamental (simple) results change if we focus on real returns. For instance, we show that in this model, when real benefits are considered the optimal stock ratio for a simple mix between a bond and a stock can heavily deviate from the corresponding nominal result (particularly for long time horizons). We also consider typical retirement savings products with and without (nominal) guarantees and analyze how their real risk-return characteristics deviate from the nominal risk-return characteristics. We particularly show that, under certain circumstances, an increase in (nominal) guarantees can increase real risk. Finally, we derive implications for consumers, financial advisors, and policy makers.

Note that in all our analyses, we analyze risk-return characteristics in real terms and still assume that – as is currently the case in practice – guarantees of typical products are given in nominal terms and that bonds with nominal notional values are used as 'safe assets' in products with balanced funds or with embedded guarantees.[5]

2 Long-term correlation between inflation and stocks

The so-called generalized Fisher-hypothesis (cf. Fisher, 1930) states (in the words of Gultekin, 1983) that 'the expected real return on common stocks and expected inflation rate vary independently so that, on average, investors are compensated for changes in purchasing power', i.e. stock/equity returns should be positively correlated with inflation. Over a rather short time horizon, however, often a negative correlation between inflation and the equity market can be observed which is sometimes referred to as 'equity return–inflation puzzle'. Practitioners often argue that this negative correlation is a consequence of central banks increasing interest rates when inflation rises making bonds comparatively more attractive leading to a decreasing demand in equities. The most famous theoretical explanation is Fama's (1981) 'proxy hypothesis'[6], stating[7] that the apparent anomalous relationship between equity returns and inflation is simply proxying the positive relationship one would expect between equity prices and real fundamentals.

Since our paper is concerned with retirement saving, i.e., saving over a time horizon that may very well stretch over several decades, we are more concerned about the long-term correlation between equity-returns and inflation. Intuitively and simplified, one would expect a positive correlation between equity markets and inflation over a long time horizon for the following reason: Picture a scenario of very low inflation over the next, say, 30 years, and a scenario of very high inflation over the same time horizon. In the low (high) inflation scenario, a company would sell its products in 30 years at rather low (high) nominal prices and would also have rather low (high) nominal expenses, e.g., for paying its employees' salaries or buying raw materials needed to produce its products. Ceteris paribus, after 30 years the ratio between the value of the company and the price of the product it produces should be the same in both scenarios. Since the ceteris-paribus-assumption is not exactly

5 The market for inflation linked bonds is not very deep. Products with guarantees in real terms can rarely be found and are to the best of our knowledge not being offered at all for regular premium payments, cf. also Graf et al. (2014).

6 testable implications of which have e.g. been confirmed by Gallagher and Taylor (2002).

7 in the words of Gallagher and Taylor (2002) who also derive and confirm testable implications of the proxy hypothesis.

fulfilled in practice, the correlation between equity returns over a long time horizon and inflation over the same time horizon will of course not be a perfect one but this intuitive argument suggests that it should be significantly above zero. Such a positive correlation **for long time-horizons** has been empirically confirmed by, e.g., Boudoukh and Richardson (1993)[8], Lothian and McCarthy (2004)[9] or Rapach (2002)[10].

If this positive correlation exists in practice, this will have important consequences for long term savings processes. E.g., an increase of the equity portion of a long-term savings product that consists of equities and bonds will (besides increasing the expected return) have two opposing effects on the risk:

1) The volatility of the terminal value increases which makes the product riskier in both, nominal, and real terms.
2) The correlation of the terminal value with long-term accumulated inflation increases, reducing the risk in real terms. Hence, the higher the correlation between long-term accumulated inflation and long-term accumulated equity returns, the larger is the difference between nominal and real risk-return indicators.

Since an increase of the equity portion has two opposing effects on real risk, it is not clear without further analysis, under which circumstances an increase in the equity ratio makes a product riskier in real terms. Since in typical guaranteed products a reduction of the guarantee level leads to an increase of the equity ratio (or more generally an increase of risky assets), the above effects also occur when the guaranteed (nominal) benefit is reduced and hence it is also not clear without further analysis, under which circumstances higher guarantee levels actually make a product 'safer' in real terms.

In the remainder of this paper, we will analyze these and related questions.

8 The authors come to the conclusion that *'In conjunction with (i) the evidence across subperiods, (ii) the consistency in results using both ex ante and ex post inflation, and (iii) the similarities using different sets of instruments, this paper provides strong support for a positive relation between nominal stock returns and inflation over long horizons'*.

9 The authors also come to the conclusion that there is a positive correlation over long time horizons: '*The puzzle therefore is not that equities fail the test as inflation hedges, as had been quite widely believed, but that they take so long to pass.'*

10 '*Overall, our results indicate that inflation does not erode the long-run real value of stocks.*'

3 Financial model

In this section, we introduce the financial model considered in our analyses. This model particularly incorporates a dependency between inflation and equity returns by applying a so-called 'cascade approach'. The subsequent Section 3.1 describes the stochastic modeling framework whereas Section 3.2 elaborates on the properties and limitations of the introduced model.

3.1 Model description

We perform our analyses in a model with stochastic interest rates, inflation, and equity returns which structurally (i.e. with respect to the relation between the equity, interest and inflation-process) coincides with the approach by Brennan and Xia (2002). As basis, we use the model that is for example used within the Austrian and the German industry standard for so-called 'products of category 4' within the abovementioned PRIIPs regime (cf. AVÖ, 2018 and DAV, 2018 and summarized by Graf and Korn, 2020).[11] This model has stochastic (nominal) interest rates applying the G2++-model (cf. Brigo and Mercurio, 2006) and further assumes a generalized Black-Scholes-model (cf. Black and Scholes, 1973) for equity returns. We expand this model by stochastic inflation using the following 'cascade approach' for the dynamics under the real-world/objective probability measure '$\mathbb{P}$':

The instantaneous inflation rate $i(t)$ follows the Vasiçek-model (cf. Vasiçek, 1977) and obeys the following stochastic dynamics

$$di(t) = a_i\big(\theta_i - i(t)\big)dt + \sigma_i dW_i(t), i(0) = i_0$$

for a $\mathbb{P}$ −Brownian motion $W_i(t)$ with $\theta_i \in \mathbb{R}, a_i \neq 0$ and $\sigma_i > 0$. Further, we apply the G2++-model (that is used for nominal interest rates in the above-mentioned industry standards) as our model for real interest rates. This model is driven by two additional stochastic processes $x(t)$ and $y(t)$ as follows:

$$dx(t) = a_x\big(\theta_x - x(t)\big)dt + \sigma_x dW_x(t), x(0) = 0$$

$$dy(t) = a_y\left(\theta_y - y(t)\right)dt + \sigma_y dW_y(t), y(0) = 0$$

[11] As previously mentioned, the PRIIP-regulation (cf. European Union, 2014) is a Pan-European regulation which requires to draw up a key information document for so-called PRIIPs (i.e. packaged retail and insurance-based investment products). For calculating the required risk/return indicators an 'industry standard'-model can be used for certain products.

where $W_x(t), W_y(t)$ are $\mathbb{P}-$Brownian motions and similar parameter restrictions for $a_x, a_y, \theta_x, \theta_y, \sigma_x$ and σ_y hold as for the specification of the inflation process. We further assume the Brownian motions to be correlated by $dW_x dW_y = \rho_{x,y}\, dt, dW_S dW_i = \rho_{S,i} dt$ and $dW_x dW_i = dW_y dW_i = 0$.

Based on the inflation rate and the real interest rate, the nominal short rate $r(t)$ then follows

$$r(t) = x(t) + y(t) + i(t) + \psi(t)$$

where $\psi(t)$ is a deterministic function to ensure that the model replicates an initial term structure of interest rates (cf. Appendix A for more details on the model and especially on the derivation of this function).

Based on $i(t)$ and $r(t)$ we introduce with $CPI(t)$ the development of accrued inflation, i.e., the development of a consumer price index and further with $C(t)$ the development of an investment in the (nominal) short rate, i.e., a bank account:

$$CPI(t) := \exp\left(\int_0^t i(s)ds\right), CPI(0) = 1$$

$$C(t) := \exp\left(\int_0^t r(s)\, ds\right), C(0) = 1$$

Finally, the equity's spot price $S(t)$ obeys the following dynamics:

$$dS(t) = S(t) \cdot \big((r(t) + \lambda_S)dt + \sigma_S dW_S(t)\big)$$

with $W_S(t)$ denoting another $\mathbb{P}-$Brownian motion with $dW_S dW_x = dW_S dW_y = 0$ and $dW_S dW_i = \rho_{S,i} dt$.

In the following analyses we will also consider equity investments $S_A(t)$ with different volatilities σ_A which are also driven by the same Brownian motion $W_S(t)$ and earn an adjusted risk premium $\lambda_A = \lambda_S \cdot \sigma_A / \sigma_S$. Hence, the dynamics of $S_A(t)$ read as

$$dS_A(t) = S_A(t) \cdot \big((r(t) + \lambda_A)dt + \sigma_A dW_S(t)\big).$$

Furthermore, the investment strategies considered below will also contain some fixed income investments and therefore we denote by $P(t,T)$ the price of a zero-coupon bond at time t with maturity $T > t$. This price is derived by means of risk-neutral valuation where we consider the above stochastic processes under the risk-neutral pricing measure '$\mathbb{Q}$' defined by setting the risk premia θ_x, θ_y and λ_S to 0 as

$$P(t,T) := \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds} | \mathcal{F}_t\right],$$

where $(\mathcal{F}_t)_t$ denotes the natural filtration generated by the Brownian motions. Appendix A.I shows that

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp(A(t,T))$$

with

$$A(t,T) = \frac{1}{2}\big(V(t,T) - V(0,T) + V(0,t)\big) - \frac{1 - e^{-a_x(T-t)}}{a_x} x(t) - \frac{1 - e^{-a_y(T-t)}}{a_y} y(t) - \frac{1 - e^{-a_i(T-t)}}{a_i}(i(t) - \theta_i) + (i(0) - \theta_i)\left(\frac{1 - e^{-a_i(T)}}{a_i} - \frac{1 - e^{-a_i(t)}}{a_i}\right)$$

and

$$\begin{aligned} V(t,T) = {} & \frac{\sigma_x^2}{a_x^2}\left((T-t) + \frac{2}{a_x} e^{-a_x(T-t)} - \frac{1}{2a_x} e^{-2a_x(T-t)} - \frac{3}{2a_x}\right) \\ & + \frac{\sigma_y^2}{a_y^2}\left((T-t) + \frac{2}{a_y} e^{-a_y(T-t)} - \frac{1}{2a_y} e^{-2a_y(T-t)} - \frac{3}{2a_y}\right) \\ & + 2\rho_{x,y}\frac{\sigma_x\sigma_y}{a_x a_y}\left((T-t) + \frac{1}{a_x}\big(e^{-a_x(T-t)} - 1\big) + \frac{1}{b_x}\big(e^{-b_x(T-t)} - 1\big) \right. \\ & \left. - \frac{1}{a_x + a_y}\big(e^{-(a_x+a_y)(T-t)} - 1\big)\right) \\ & + \frac{\sigma_i^2}{a_i^2}\left((T-t) + \frac{2}{a_i} \mathrm{e}^{-a_i(T-t)} - \frac{1}{2a_i} \mathrm{e}^{-2a_i(T-t)} - \frac{3}{2a_i}\right). \end{aligned}$$

In the above pricing formula, $P^M(0,t)$ denotes the price of a zero-coupon bond at time 0 with a maturity of t years observed in the market (i.e., the initial term structure of interest rates). In our modelling approach, this term structure of interest rates is specified by a Nelson-Siegel-Svensson approach (cf. Svensson, 1994) who postulate the spot rate $z(0,t)$ as

$$z(0,t) = \frac{1}{100} \cdot \left(\beta_0 + \beta_1\left(1 - e^{-\frac{t}{\tau_1}}\right)\frac{\tau_1}{t} + \beta_2\left(\left(1 - e^{-\frac{t}{\tau_1}}\right)\frac{\tau_1}{t} - e^{-\frac{t}{\tau_1}}\right) + \beta_3\left(\left(1 - e^{-\frac{t}{\tau_2}}\right)\frac{\tau_2}{t} - e^{-\frac{t}{\tau_2}}\right)\right).$$

We take the required parameters $\beta_0, \beta_1, \beta_2, \tau_1$ and τ_2 from the German federal reserve bank following Schich (1997) who then sets the corresponding zero-coupon bond prices as

$$P^M(0,t) = \left(1 + z(0,t)\right)^{-t}.$$

3.2 Properties and limitations of the cascade style model

3.2.1 Properties

Note that the ‘cascade structure’ of the processes used in our model assumes that the drift parameter of the equity process depends on the nominal short rate (and thus inflation). This leads to the desired positive correlation between long-term inflation and long-term equity returns – even if one assumed a (moderately) negative correlation $\rho_{S,i}$ between W_i and W_S and hence a negative (instantaneous) short-term-correlation between inflation and equity returns.

To illustrate this effect, Figure 1 depicts the correlation coefficient, i.e. $Corr\left(\ln\left(S_A(T)\right), \ln\left(CPI(T)\right)\right) := \frac{Cov\left(\ln\left(S_A(T)\right), \ln\left(CPI(T)\right)\right)}{\sqrt{Var\left(\ln\left(S_A(T)\right)\right) \cdot Var\left(\ln\left(CPI(T)\right)\right)}}$, of log-returns $\ln S_A(T)$ of an equity investment for different volatilities and the long-term cumulated (logarithmic) inflation rate $\ln CPI(T)$ by considering an investment horizon of $T = 30$ years. In this figure, we assume the parameter set underlying our numerical analyses and summarized in Table 1 in Section 5.2 and calculate the correlation coefficient for different assumptions on the instantaneous correlation $\rho_{S,i}$ by setting $\rho_{S,i}$ to $-100\%, -50\%, -10\%, 0\%, 10\%, 50\%, 100\%$.[12]

[12] Cf. Appendix A.II for an analytical derivation of the considered correlation coefficient.

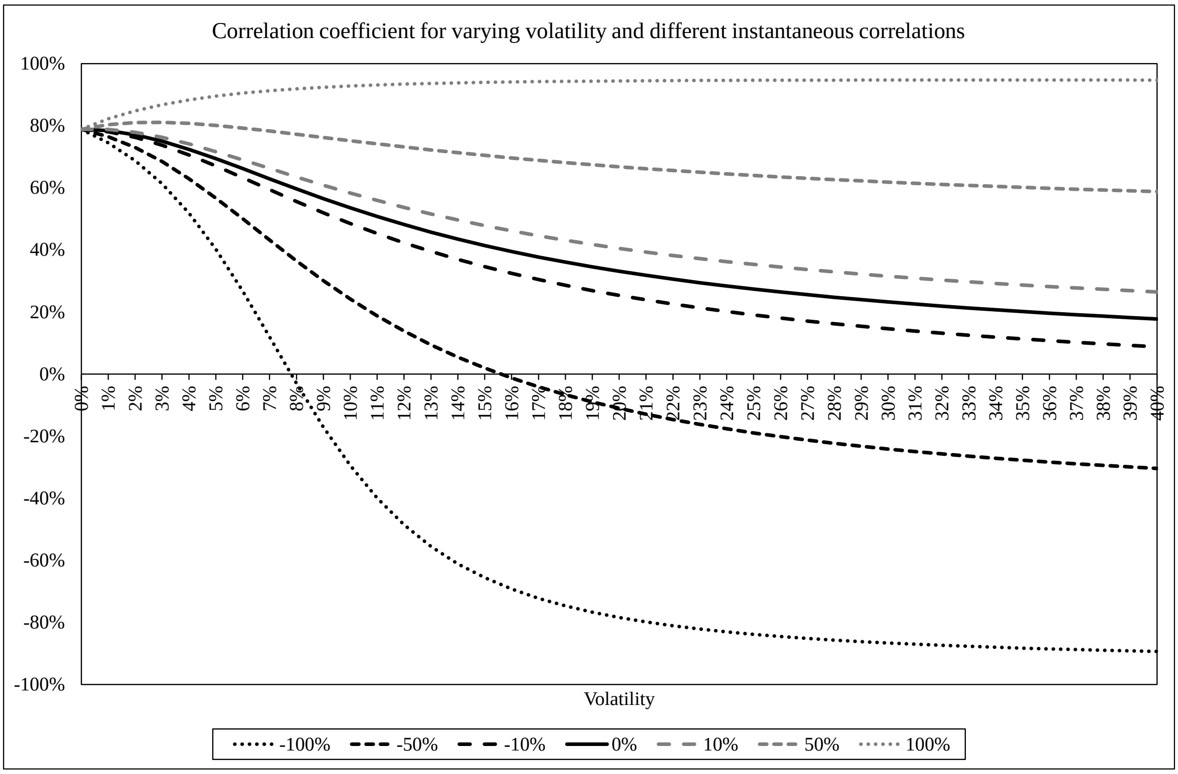

Figure 1 Model-implied correlation of equity returns and inflation over an investment horizon of 30 years for different volatilities of the underlying equity investment and different instantaneous correlation coefficients $\rho_{S,i}$

We can conclude that for reasonable choice of parameters – in particular if the instantaneous correlation $\rho_{S,i}$ is set to a moderately negative value as sometimes observed in practice or to 0% as in the analyses below – our modeling approach yields the empirically observed positive correlation between equity returns and inflation over the long run. For rather low equity volatilities, the positive long-term correlation between equity and inflation even persists for a highly negative instantaneous correlation.

We also observe that long-term correlation between equity returns and cumulated inflation decreases as the volatility of the underlying equity investment increases. Hence, although we incorporate the rate of inflation into the equity's expected return, this model-implied correlation can be overweighed by random fluctuations of the equity investment – which in our view makes sense from an economical point of view. The equity investment is expected to grow 'in line' with the realized inflation over the long run, but also bears some 'inflation-independent risk' which is higher for more volatile equity investments.

If we consider a shorter investment horizon of $T = 5$ years as depicted in Figure 2, we observe significantly lower correlations in line with the empirical observations cited in Section 2.

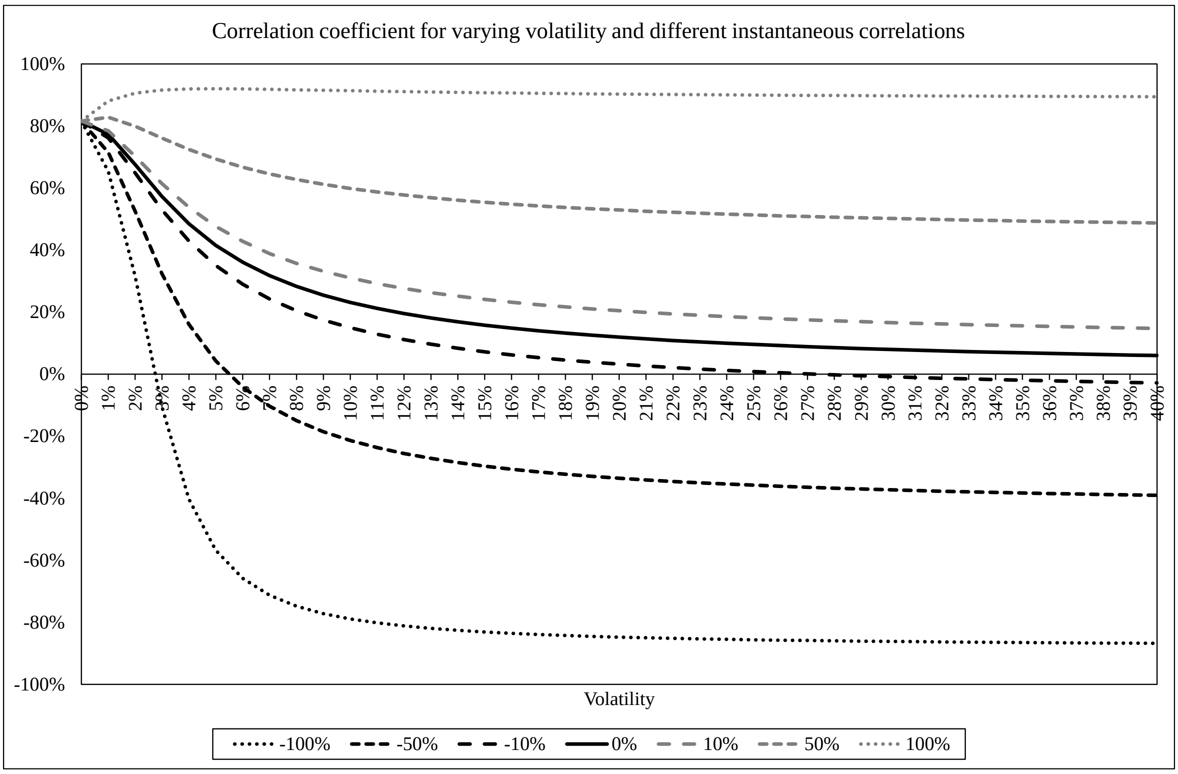

Figure 2 Model implied correlation of equity returns and inflation over an investment horizon of 5 years for different volatilities of the underlying equity investment and different instantaneous correlation coefficients $\rho_{S,i}$

3.2.2 Limitations

Figure 1 also reveals an undesired property of our model. As the equity's volatility approaches zero, we observe that the resulting correlation coefficient increases. This implies that a pure money market investment (with no volatility) provides the 'best' inflation hedge in our model which clearly contradicts economic results and the more fundamental interactions underlying the empirical results of Section 2. From an economic point of view, the positive correlation of equity returns and inflation stems from the fact that future profits and hence dividends are supposed to grow 'in line' with realized inflation which equity returns should reflect. Since we do not model these interactions of dividends and subsequent valuations of equities explicitly, but rather implicitly by the introduced approach, we get the undesired side-effect that our model may yield implausible results when the volatility is reduced 'too much' and hence the considered instruments transform from an equity investment (where the interactions, esp. dividends can actually be observed) to a non-dividend paying asset which purely resembles a money market investment and hence the economic interactions will not apply. Hence, our model (just like all models) may be a suitable choice to analyze certain questions (particularly related to real risk-return characteristics of long-term savings processes with 'risky' assets) and a poor choice for other questions.

3.2.3 Consequences

In summary, our model proves to be a good compromise between ease of use and proper implementation of underlying economic effects. Nonetheless, caution has to be taken when money-market investments or equity investments with very low volatility are considered, since then the inflation-adjusted results can be misleading. One example is that the famous Merton optimization problem (cf. Merton, 1969) yields the same optimal asset allocation between equity investments and the money-market, for both, a nominal or an inflation adjusted view (cf. Appendix B for the corresponding derivation). This results from the fact, that in the model, the equity investment and the money market account provide the same inflation-protection and only differ by their volatility.

We would like to stress that in the following analyses, no money market accounts are considered. Further, we only show results for equity-volatilities of at least 5% where in our view the model appears suitable.

4 Analysis of a simple investment strategy

In this section we analyze a simple investment strategy which is suitable to illustrate the structural differences that occur when utility is derived from real as opposed to nominal values. Moreover, despite its simplicity this strategy is of very high practical relevance:[13] We assume that the client's single premium[14] at $t = 0$ is split into an investment in a zero-coupon bond with maturity T and an equity investment. Hence, for an equity ratio of α, the client's account value $A_\alpha(T)$ at maturity is given by

$$A_\alpha(T) = (1-\alpha)\frac{1}{P(0,T)} + \alpha\frac{S_A(T)}{S_A(0)}$$

[13] We will see in Section 5 that this investment strategy is the basic building block in many unit-linked products with (nominal) maturity guarantee that are offered for retirement saving.

[14] Without loss of generality, we assume a single premium of 1 in this setting.

where $P(0,T)$ denotes the price at time 0 of a zero-bond paying 1 unit of currency at time T and $S_A(T)$ denotes the spot price at time T of an equity investment equipped with volatility σ_A. As a consequence, this static investment strategy comes with a (nominal) guarantee of $(1-\alpha)\frac{1}{P(0,T)}$ at maturity.

The inflation-adjusted value $\tilde{A}_\alpha(T)$ at maturity is given by

$$\tilde{A}_\alpha(T) := \frac{A_\alpha(T)}{CPI(T)} = (1-\alpha)\frac{1}{P(0,T)\cdot CPI(T)} + \alpha\frac{S_A(T)}{S_A(0)\cdot CPI(T)}.$$

First, we illustrate the risk-return characteristics of this investment strategy assuming the capital market assumptions as summarized in Table 1 in Section 5.2.

The left part of Figure 3 shows the nominal respectively real risk return characteristics of this investment strategy for different values of the equity ratio where volatility (i.e. a symmetric measure) is used as a risk measure (and expected return as a return measure). We observe that the expected real return is always lower than the expected nominal return. Real risk is however lower than nominal risk for higher equity ratios and higher than nominal risk for lower equity ratios. Nevertheless, within each 'dimension' (i.e., nominal or real), we always observe that an increase in the equity ratio leads to an increase in both, risk and return.

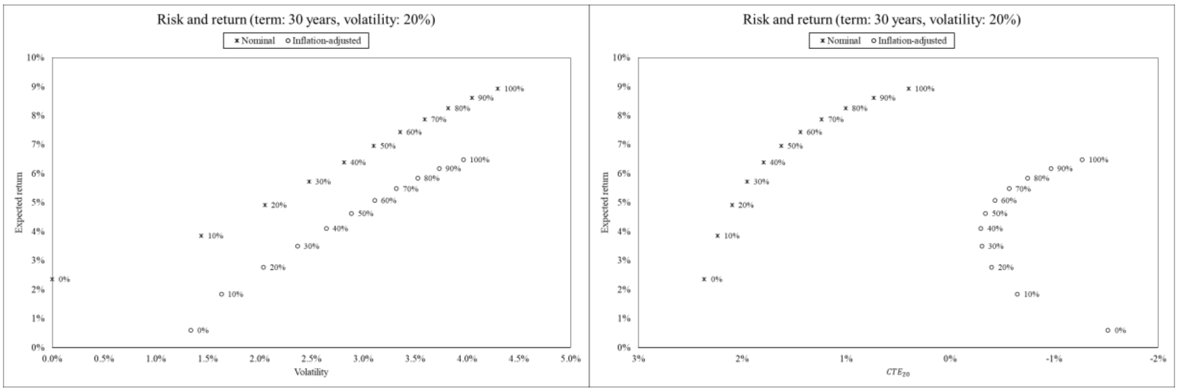

Figure 3 Nominal and inflation-adjusted risk-return characteristics for the simple investment strategy with $\alpha = 0, \dots, 100\%$ (left: volatility, right: CTE_{20})

This changes dramatically if we use a 'downside based'-risk measure. The results for 'CTE_{20}'[15] as a risk measure are displayed in the right part of Figure 3. Here, nominally we again observe that an increase in the equity ratio leads to an increase in both, risk and return.

[15] i.e., the conditional tail expectation to a level of 20%.

However, in real terms we observe a remarkably different structure: Risk is minimal for an equity ratio of roughly 40%. Above this ratio, an increase in the equity ratio leads to an increase in both, risk and return. However, if the equity ratio is reduced below 40%, a reduction in the equity ratio leads to a reduction in return and an increase in risk.

While these results can intellectually be easily understood as a consequence of the positive long-term correlation between equities and inflation, they have massive implications on retirement planning: If based on nominal risk-return characteristics an equity ratio is selected that matches a consumer's risk aversion, the product's relevant real risk-return characteristics might be completely unsuitable for this consumer.

Next, we analyze the expected utility resulting from an investment in these simple investment strategies. We will measure utility by a CRRA-utility function $u_\gamma(x)$

$$u_\gamma(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma \neq 1 \\ \log(x), & \gamma = 1 \end{cases}$$

with risk aversion parameter γ and will then maximize $\mathbb{E}[u_\gamma(A_\alpha(T))]$ and $\mathbb{E}\left[u_\gamma\left(\tilde{A}_\alpha(T)\right)\right]$ with respect to the equity ratio α. Since the considered random variables follow a 'shifted' lognormal distribution (i.e., a sum of log-normally distributed random variables), no closed form solutions for the underlying probability distributions are known. Therefore, for deriving the equity ratio α which maximizes the expected utility $\mathbb{E}\left[u_\gamma[.]\right]$ for $\alpha \in [0,1]$, we rely on a numerical approach briefly sketched as follows:

(1) We solve for the root of the (numerically approximated) first derivative of $\mathbb{E}\left[u_\gamma[.]\right]$ for fixed γ as a function of the equity ratio α for $\alpha \in [0,1]$.

If such a root $\alpha^\star$ exists in $[0,1]$ we conclude that $\alpha^\star$ maximizes the expected utility, since $u_\gamma[.]$ is a concave function.

(2) If a root of the derivative cannot be found within the interval $[0,1]$ we compute the expected utility for $\alpha = 0$ and $\alpha = 1$ and accordingly set the utility maximizing equity ratio as either 0 or 1.

$\mathbb{E}\left[u_\gamma[.]\right]$ is approximated by means of Monte-Carlo simulation (similar with our numerical analyses in Section 5) of the underlying stochastic processes. The required first derivative of $\mathbb{E}\left[u_\gamma[.]\right]$ interpreted as a function of the equity ratio α is then approximated by finite differences. Finally, we use a Brent-solver to solve for the corresponding root (cf. Brent, 1973).

The following results illustrate how the optimal (utility maximizing) equity ratio structurally changes when real (rather than nominal) returns are considered. We particularly analyze the impact of term to maturity and start with a short term of 5 years: Figure 4 shows the optimal equity ratio as a function of equity-volatility for different levels of risk aversion for a term of $T = 5$ years.

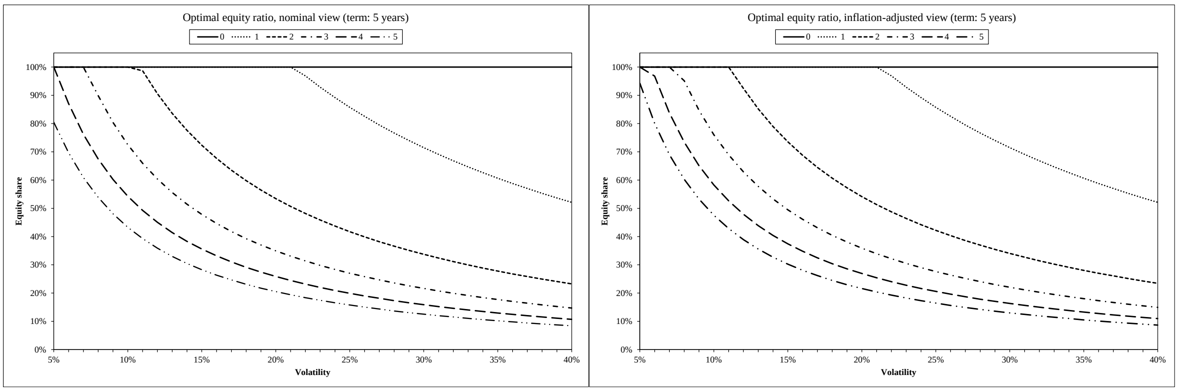

Figure 4 Optimal equity ratio for nominal (left) and inflation-adjusted view (right) of the static strategy for a term of 5 years for different risk aversion parameters $\gamma = 0, 1, \ldots, 5$.

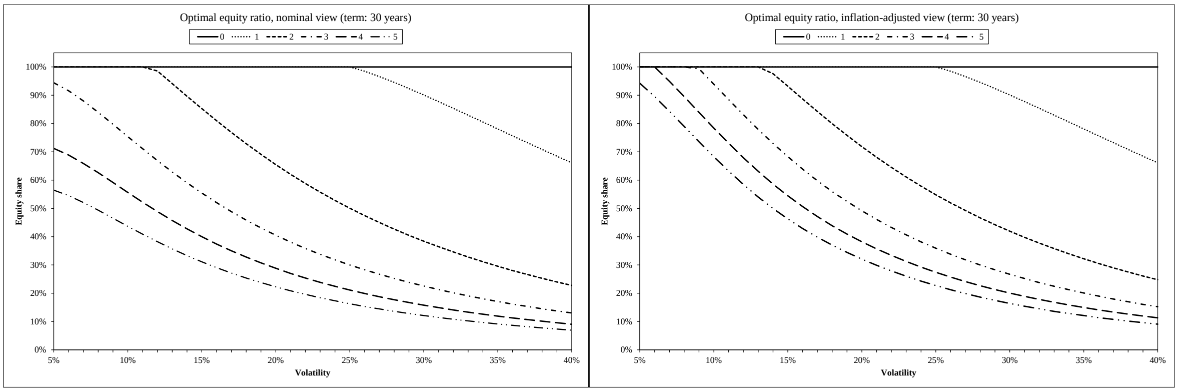

Figure 5 Optimal equity ratio for nominal (left) and inflation-adjusted view (right) of the static strategy for a term of 30 years for different risk aversion parameters $\gamma = 0, 1, \ldots, 5$.

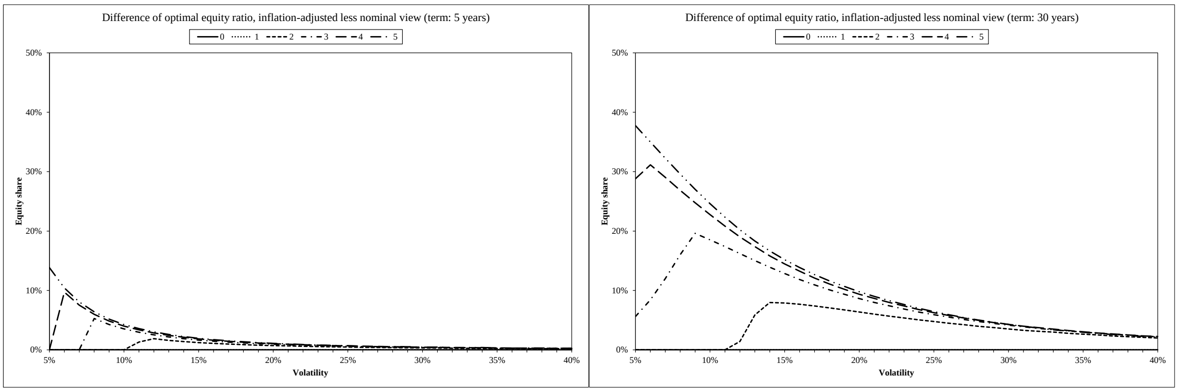

Figure 6 Difference of optimal equity ratio in percentage points (inflation-adjusted less nominal one) for a term of 5 years (left) and 30 years (right) for different risk aversion parameters $\gamma = 0, 1, \dots, 5$

The left chart of Figure 4 assumes that utility is drawn from nominal values, the right chart takes an inflation-adjusted point of view. The left chart of Figure 6 shows the difference between the optimal equity ratio under real respectively nominal terms.

As expected, we observe that the optimal equity ratio is in general decreasing in volatility and risk aversion. Since for a rather short investment horizon of 5 years, the correlation between equity returns and inflation (and hence the 'inflation protecting effect' resulting from an equity investment) is rather moderate (cf. Figure 2), volatility is the dominating risk in this setting. Hence, the optimal equity ratios in the nominal and real setting are very similar unless for a very low volatility, where the relevant correlation is higher and hence a higher equity ratio is preferred. This is, however, primarily the case for higher levels of risk aversion, since otherwise for low volatility an equity ratio of 100% is optimal in both, nominal and real terms.

Figure 5 shows the same results as Figure 4, i.e., the utility-maximizing equity ratio as a function of equity volatility and for different levels of risk aversion, but now for a term of $T = 30$ years. From this figure and the right panel of Figure 6 which depicts the differences in the resulting optimal equity ratios in percentage points, we observe that the optimal equity ratio is again decreasing in volatility and risk aversion. But for a longer time horizon, the transition from a nominal to a real viewpoint has a much larger effect than for the shorter horizon above (confirming our claim that this topic is particularly relevant for long-term savings processes, e.g. in the context of retirement savings). The optimal equity ratios in real terms are significantly higher than in nominal terms. This now also holds for higher volatility and for all levels of risk aversion > 1. Particularly, for consumers with a rather high risk aversion ($\gamma = 3, 4, 5$), up to a volatility of 20%, the optimal equity ratio in real terms is more than 10 percentage points higher than an optimization in nominal terms would indicate.

Only for very high volatilities, i.e., when random fluctuations of the equity investment and not inflation are the main driver for risk, the nominal and real view lead to very similar results.

Finally, we would like to mention that for $\gamma = 0$ as well as for $\gamma = 1$, the optimal equity ratio does not change if we move from nominal to real returns. For $\gamma = 0$, an equity ratio of 100% is always optimal[16] and for $\gamma = 1$ it is easy to show that $\mathbb{E}\left[u_\gamma\left(A_\alpha(T)\right)\right] = \mathbb{E}\left[u_\gamma\left(\tilde{A}_\alpha(T)\right)\right] + c$ for some constant c.

In summary, our results clearly show that (for long time horizons) different investment strategies need to be selected for reducing real risk than for reducing nominal risk. In particular, we can conclude that for long term savings processes in general higher equity ratios should be chosen when risk in real terms is considered the relevant risk. We will see in Section 5 that this implies for typical old-age provision products that lower guarantees should be chosen when risk in real terms is considered the relevant risk. Also, the topic is particularly relevant for investors with a rather high degree of risk aversion. While investors with a rather low degree of risk aversion primarily maximize expected return (which is accomplished by essentially the same strategy no matter if real or nominal return is considered), investors with a rather high degree of risk aversion in contrast put more focus on reducing risk.

These results nicely relate to the two opposing effects on risk explained in Section 2: The shorter the investment horizon and the higher the volatility the more relevant is the risk of random fluctuations which impacts nominal and real risk alike. If, however, the investment horizon is long and volatility is not too high, the uncertainty of inflation becomes a main driver of equity performance which increases nominal but decreases real risk of equity investment.

5 Nominal and inflation-adjusted risk-return profiles for typical old-age provision products

In this section we derive results for (stylized) products that are typically offered for old-age provision. Our analyses will be carried out in the capital market model introduced in Section 3 and we will use Monte-Carlo simulation to derive our results. In particular, we assume a daily simulation of the underlying processes and assume 21 trading days per month. Hence,

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16 Note, we did not consider any leverage to potentially increase the equity ratio above 100%.

the step size of our simulation exercise is given by $\Delta t = \frac{1}{12 \cdot 21}$ and we further generate 10.000 trajectories.

5.1 Considered products

We consider three types of products that will be explained in more detail in the remainder of this subsection:

(1) a balanced fund investing a certain equity ratio in an equity investment and the rest in a zero-coupon bond (using daily rebalancing),

(2) a guarantee product generating the guarantee by implementing a dynamic CPPI-strategy[17] (on a daily and client-individual basis) and

(3) a static guarantee product where a zero coupon bond is held to maturity to generate a certain guarantee and the rest is invested into an equity fund.[18]

We focus on the savings process with a term to maturity of T years and assume that each product pays a maturity benefit at time T which would typically be close to the client's retirement date. We consider products with a single premium P and neglect any charges.

With $A(t)$ denoting the client's account value at time t and $Perf_{t,t+\Delta t}$ denoting the performance of the considered products from t to t+Δt, we set $A(0) = P$ and project the client's account value by

$$A(t + \Delta t) = A(t) \cdot Perf_{t,t+\Delta t}.$$

For the guarantee products, we consider different levels l of guarantees by defining the maturity guarantee $G_T(l)$ at time T as $G_T(l) = l \cdot P$.

Balanced funds

For this product, the premium is invested into a balanced fund which invests partly into a zero-coupon bond with duration T and partly in equity. The duration of the bond matches the duration of the contract. We assume that the equity ratio of the balanced fund is readjusted on a daily basis. Hence, for an equity ratio of α, the daily performance of the client's account value $A(t)$ is given by

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17 Cf. Black and Perold (1992).

18 The three products are similar to those analyzed in Graf et al. (2012) which contains more details on the modeling approach. Note that the third product is essentially the simple investment strategy analyzed in Section 4.

$$Perf_{t,t+\Delta t} = \left((1-\alpha) \frac{P(t+\Delta t, T)}{P(t,T)} + \alpha \frac{S_A(t+\Delta t)}{S_A(t)} \right).$$

Dynamic guarantee product (I-CPPI)

In this product, the well-known CPPI-algorithm allocating money to a riskless asset (in our case a zero-coupon bond) and a risky asset (in our case equity) is applied on a client-individual basis. In theory, the asset allocation of CPPI-products is adjusted continuously according to some given rule. In practice however, such re-allocations can only be applied at certain points in time. In our numerical analysis (as typically also in practice) this is done on a daily basis. At each trading date the provider determines the present value of the guarantee (so-called floor) $F(t) \coloneqq G_T(l) \cdot P(t,T)$ and invests a multiple m of the so-called cushion $\big(A(t) - F(t)\big)$ in the underlying equity investment. Therefore, this CPPI rule defines a path-dependent equity ratio $\alpha(t)$ by

$$\alpha(t) = \frac{\max\Big(0, \min\Big(A(t), m \cdot \big(A(t) - F(t)\big)\Big)\Big)}{A(t)}.$$

The daily performance of the client's account value $A(t)$ is then given by

$$Perf_{t,t+\Delta t} = \left((1-\alpha(t)) \frac{P(t+\Delta t, T)}{P(t,T)} + \alpha(t) \frac{S_A(t+\Delta t)}{S_A(t)} \right).$$

Obviously in practice – when continuous rebalancing is impossible – the product provider faces two sources of risk within a CPPI structure: First, the risky asset might lose more than $\frac{1}{m}$ during one period (this risk is often referred to as gap risk or overnight risk). Second, the floor might have changed within one period due to interest rate fluctuations. Since we perform analyses from a client's perspective, we do not investigate how the product provider deals with these risks[19]. Note, in case $A(t) < F(t)$ the definition of $\alpha(t)$ ensures that at most the currently available amount $A(t)$ is invested in the riskless asset and the product can be 'underhedged', i.e. the product provider's will realize a loss at the contract's maturity.

Static guarantee product (zero + underlying)

This simple but in many markets highly relevant product also invests in a riskless asset (in our case a zero-coupon bond) and a risky asset (in our case equity). At the start of the contract, the allocation in the riskless asset is determined by investing the present value of

[19] Cf. Graf et al. (2012) and references therein for more details.

the guarantee, i.e. $F(t)$, into the riskless asset. The remainder of the premium is then invested in the risky asset and no future reallocations will be performed. Hence, the equity ratio is then given by:

$$\alpha(t) = \frac{\max\Big(0, \min\Big(A(t), \big(A(t) - F(t)\big)\Big)\Big)}{A(t)}$$

Obviously, this product is a special case of the dynamic guarantee product with a multiplier of $m = 1$. The daily performance of the client's account value $A(t)$ is again given by

$$Perf_{t,t+\Delta t} = \left((1 - \alpha(t))\frac{P(t + \Delta t, T)}{P(t, T)} + \alpha(t)\frac{S_A(t + \Delta t)}{S_A(t)}\right).$$

5.2 Numerical Results

In this section, we show the results of our analyses for the products described in the previous section.

5.2.1 Assumptions

In Table 1, we summarize the capital market parameters used in our base case.

Parameter	Value	Parameter	Value
a_i	10%	β_0	0.32994
θ_i	2%	β_1	-1.10840
σ_i	1%	β_2	-1.01065
$i(0)$	2%	β_3	0.05417
a_x	39.120%	τ_1	2.93927
σ_x	1.239%	τ_2	0.64206
θ_x	-0.330%	λ_S	4%
a_y	7.850%	σ_S	20%
σ_y	0.832%	$\rho_{S,i}$	0
θ_y	2.550%	$\rho_{x,y}$	-64.500%

Table 1 Capital market parameters (base case)

Furthermore, we consider

- equity ratios of $\alpha \in \{0\%, 25\%, 50\%, 75\%, 100\%\}$ for the balanced fund,
- guarantee levels of $l \in \{50\%, 60\%, 70\%, 80\%, 80\%, 100\%, 104\%\}$ for both guarantee products, and
- a multiplier of $m = 5$ for the I-CPPI product.

Note that under the term structure of interest rates at the start of the contract, a level of $l = 104\%$ is the maximum possible guarantee that can be provided in the base case.

5.2.2 Nominal risk and return of the products

Figure 7 shows the nominal risk and return potential of the considered products for a term of 30 years using the ‘downside based’-risk measure CTE_{20} introduced in Section 4.

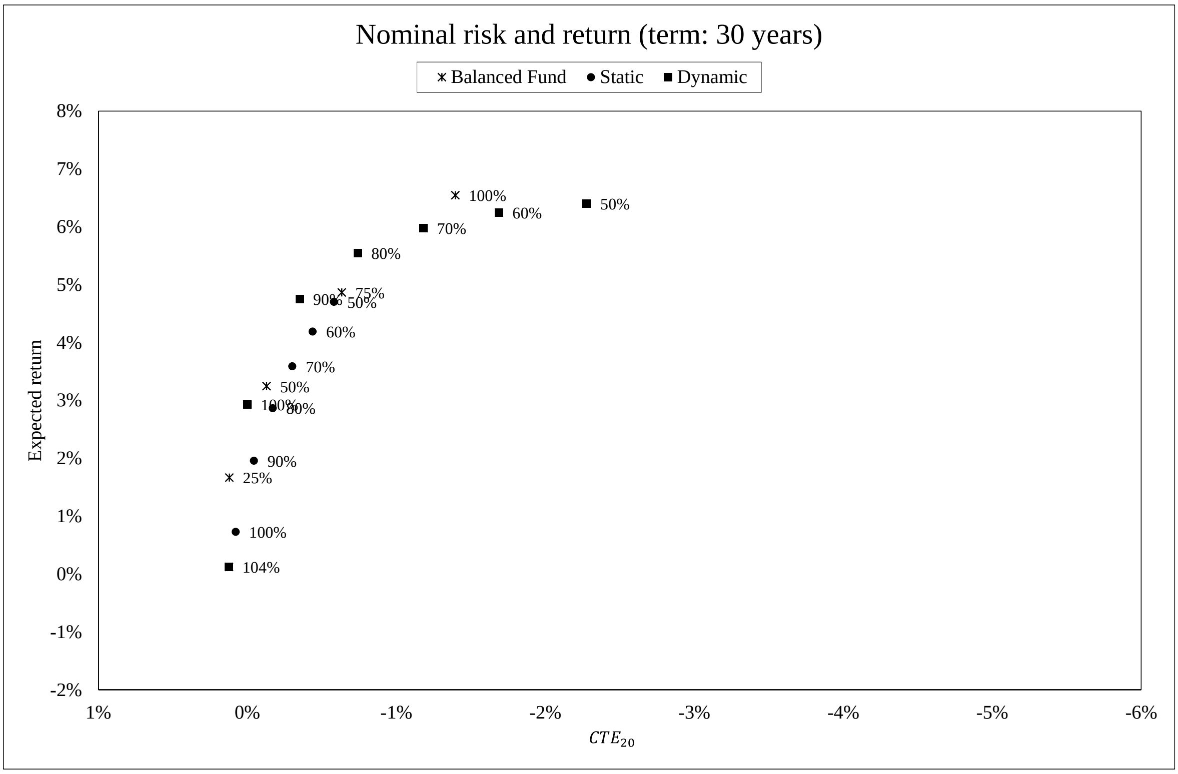

Figure 7 Nominal risk and return potential for different equity ratios of the blanced fund and different guarantee levels of the guarantee products

Under nominal terms, the risk and return potential of the products show an intuitive and somewhat expected pattern. For a guarantee level of 104%, both guarantee products consist of a 100% zero coupon investment and thus coincide with the balanced fund with an equity ratio of $\alpha = 0$. This product comes with the lowest return potential and almost the lowest risk.[20] Increasing the equity ratio of the balanced fund, or decreasing the guarantee level of the guarantee products, at the same time increases the return potential and the risk of the products.

All products show a convex pattern, i.e. the incremental increase of return is getting smaller and smaller as we decrease the guarantee (respectively increase the equity ratio of the balanced fund) by the same amount (e.g. from a guarantee level of 100% to 90%, from 90% to 80%, etc.) whereas the increase of incremental increase of risk is getting larger and larger.

For the dynamic guarantee product, once the guarantee level has been reduced to $l = 70\%$, further reductions in the guarantee level only slightly increase the return potential of the product. However, the risk of the product still increases with further reductions of the guarantee level. For a guarantee level of $l = 50\%$ and the considered risk measure, the risk of the I-CPPI product is even higher than for a pure equity fund (balanced fund with $\alpha = 100\%$). The main reason is that (although a guarantee level of $l = 50\%$ might seem rather low) the I-CPPI product requires an investment in the safe asset during the term of the contract in some simulation paths (either due to poor performance of the risky asset or due to an increase in the floor triggered by falling interest rates). If in some of these paths the risky asset increases again, the I-CPPI product participates in this increase to a lesser extent than the pure equity fund. Hence, in some of these scenarios, the pure equity fund can show a higher return than some of the considered guarantee products. Note however, that this is an effect of the considered risk measure CTE_{20}. If we move further in the distributions' tail (cf. analyses in Section 5.2.4.3), we observe that the guarantee products show a lower risk than the pure equity fund.

5.2.3 Inflation-adjusted risk and return of the products

Figure 8 shows the inflation-adjusted ('real') risk and return potential of the same products that had been displayed in Figure 7.

[20] Note that the balanced fund with an equity ratio of $\alpha = 25\%$ results (due to diversification) in a slightly lower risk than the pure investment in the risk-free investment, cf. also footnote 4.

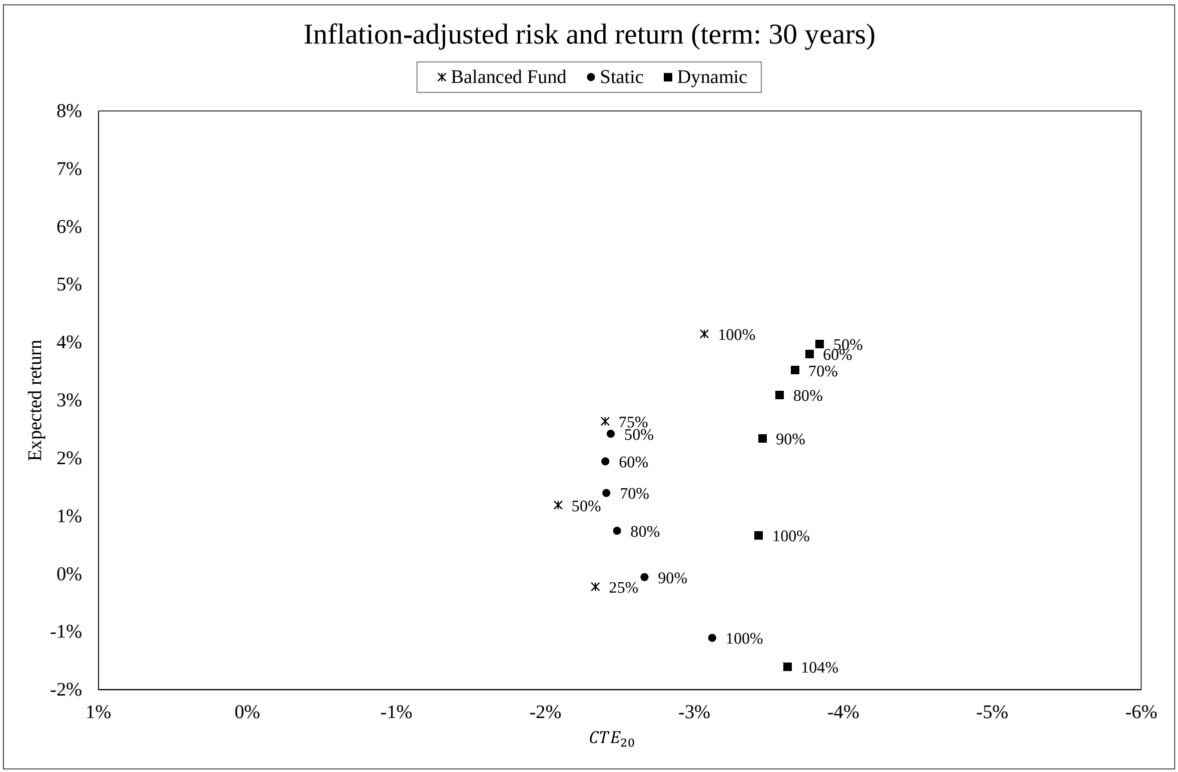

Figure 8 Inflation-adjusted risk and return potential for different equity ratios of the blanced fund and different guarantee levels of the guarantee products

Under real terms, the effect of different guarantee levels / equity ratios on the return potential of the products is similar to the one observed under nominal terms: Reducing the guarantee level of the guarantee products (or increasing the equity ratio of the balanced fund) always leads to an increase in the return potential. The increases are similar to those observed under nominal terms and the effect decreases for lower guarantee levels or higher equity ratios.

The effect on the products' real risk, however, is completely different from the observed effect on nominal risk. In particular, we find that guarantees can be 'too high', i.e. above a certain 'critical guarantee level', a further increase of the guarantee can reduce real return while at the same time increasing real risk. This happens, e.g. when the guarantee level is increased from $l = 90\%$ to $l = 104\%$ in either of the guaranteed products. We observe the same pattern for a decrease in the equity ratio of the balanced fund from $\alpha = 50\%$ to $\alpha = 25\%$: The product becomes more risky while at the same time it has a lower expected return.

For both guarantee products, the guarantee level showing the lowest risk under the considered risk measure, is significantly below the maximum possible guarantee. For the dynamic guarantee product, the effect of the guarantee level on the product's risk in real terms is very low. Even products with a guarantee level of $l = 50\%$ or $l = 60\%$ show a similar risk as products with high guarantee levels.

The observed effects can be explained by the two opposing effects on real risk explained in Section 2: When the equity ratio of the products is increased, risk stemming from the volatility of the equity investment is increased, but, at the same time, inflation risk is reduced. As a consequence, even for risk averse consumers, the product with the highest guarantee in general is not the optimal product after inflation has been taken into account. Product versions with very high guarantee levels[21] are in this setting 'dominated' by products with reduced guarantee levels in real terms under the considered risk and return measures.

5.2.4 Sensitivity Analyses

In this section, we vary the underlying interest rate environment in Section 5.2.4.1, consider the impact of a lower volatility of the underlying equity investment in Section 5.2.4.2 and investigate the impact of different risk measures in Section 5.2.4.3.

5.2.4.1 Environment of higher interest rates

In a first step, we analyze the impact of a higher interest rate environment on our results. The base case assumption set was calibrated to very low (even negative) interest rates prevailing in Germany at the end of 2021. As a sensitivity, we now assume an interest rate environment similar to that observed by the end of 2022. In 2022, after a long period of low interest rates, a hike in interest took place in many countries worldwide mainly explained by many central banks increasing their short term rates in order to cope with the globally observed spike in inflation.

Table 2 summarizes the parameter set for the initial term structure of interest rates in this section.[22] In addition, Figure 9 depicts the interest rates (in terms of spot rates) that feed into our numerical analyses for the base case and the sensitivity analysis, respectively.

[21] 'Very high' in this context always needs to be assessed in line with the maximum possible guarantee that depends on the prevailing term structure of interest rates.

[22] Note, in comparison to the base case parameter set (cf. Table 1) we only modify those parameters with respect to the initial term structure of interest rates, i.e. $\beta_0, \beta_1, \beta_2, \tau_1$ and τ_2.

Parameter	β_0	β_1	β_2	β_3	τ_1	τ_2
Value	0.90536	0.08032	4.34532	4.8861	0.82692	10.61343

Table 2 Capital market assumptions in sensitivity analysis (only initial term strucutre of interest rates)

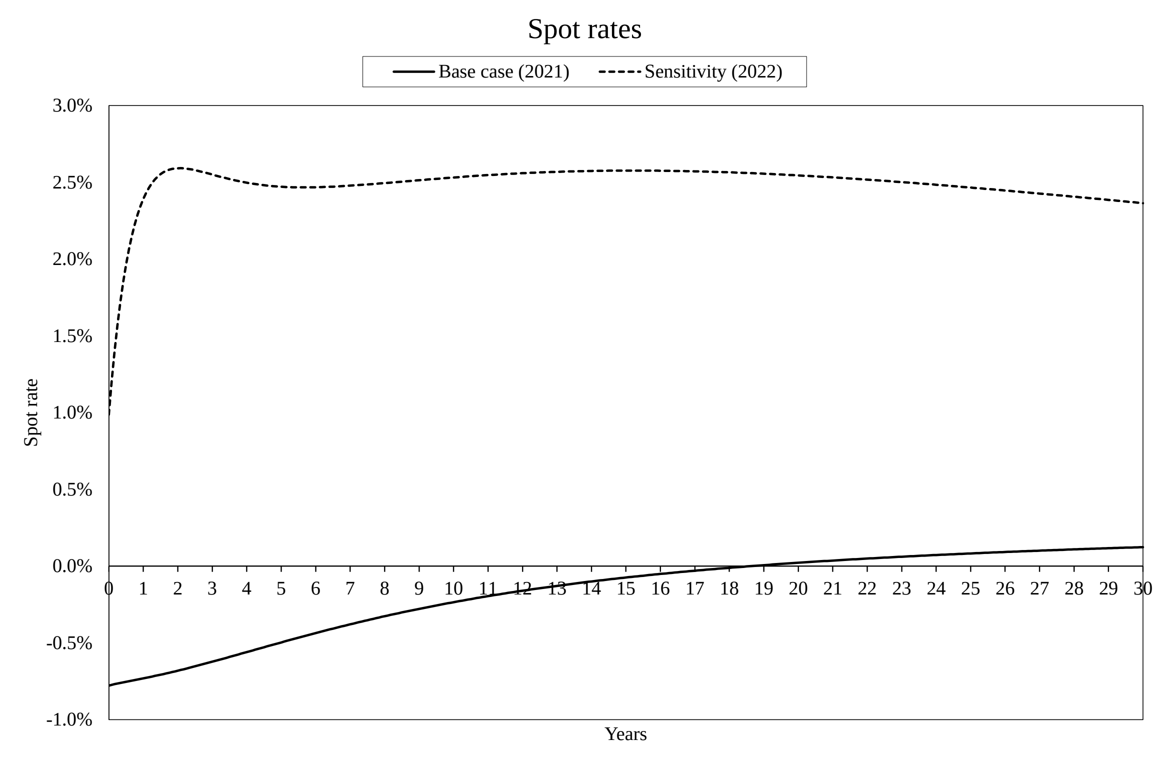

Figure 9 Interest rates as prevailing in the base case (end of 2021 in Germany) and the considered sensitivity (end of 2022 in Germany)

Since the higher interest rate environment in this analysis allows for a higher maximum possible guarantee (202% for a term of 30 years), we analyze guarantee levels $l \in \{100\%, 120\%, 140\%, 160\%, 180\%, 200\%, 202\%\}$ in what follows.

Figure 10 shows the nominal and inflation-adjusted ('real') risk and return potential of the considered products in case of a higher interest rate environment.

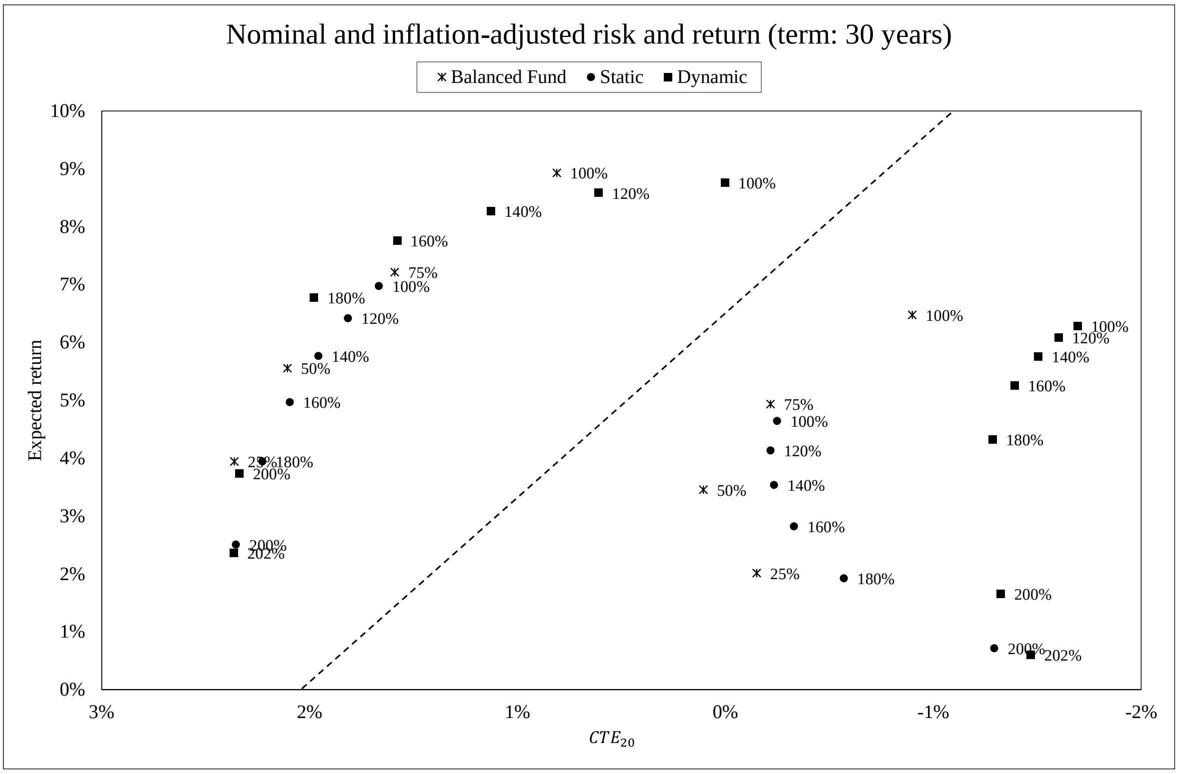

Figure 10 Nominal (upper-left part) and inflation-adjusted (lower right part) risk and return potential in case of a higher interest rate environment

From a qualitative viewpoint, the results under a higher interest rate environment show the same pattern as in the base case (cf. Figure 7 and Figure 8). The nominal results (upper-left part of the chart) reassure the 'common sense' that an increase in guarantee levels c.p. reduces the expected return, but similarly also the products' risk. For the inflation-adjusted risk-return profiles however, reducing the guarantee level of the guarantee products (or increasing the equity ratio of the balanced fund) still leads to an increase in the return potential but not necessarily on the real risk. As in the base case, we observe that for guarantees above a certain level, a further increase of the guarantee can reduce real return while at the same time increasing real risk – however due to the higher interest rates this 'critical guarantee level' is now much higher.

It is known form behavioral economics that due to loss aversion, products with a guarantee level of less than 100% appear very unattractive to many consumers.[23] Also, sometimes a guarantee level of 100% is required by the legislator.[24] Our results clearly show that when

[23] Cf., e.g., Dichtl and Drobetz (2011), Ebert et al. (2012) or Ruß and Schelling (2018) and references therein.

[24] This is, e.g., the case with the so-called Riester-Rente, a government subsidized pension product in Germany.

interest rates are rather low (as in our base case) a guarantee level of 100% can be objectively too high (even for risk averse consumers) since it comes with a lower expected (real) return and a higher (real) risk than products with a lower guarantee. In contrast, in an environment of rather high interest rates, a guarantee level of 100% delivers a risk-return profile that is suitable for some risk aversion also under the inflation-adjusted view.

5.2.4.2 Lower equity volatility

In this section, we consider the impact of a lower volatility of the underlying equity investment by assuming $\sigma_S = 15\%$ instead of $\sigma_S = 20\%$. Note, all other parameters remain unchanged. In particular, we leave the equity risk premium $\lambda_S = 4\%$ and hence implicitly increase the sharpe ratio from $0.2 = \frac{4\%}{20\%}$ to $0.27 = \frac{4\%}{15\%}$.

Figure 11 shows the nominal and inflation-adjusted ('real') risk and return potential of the considered products in case of a lower equity volatility.

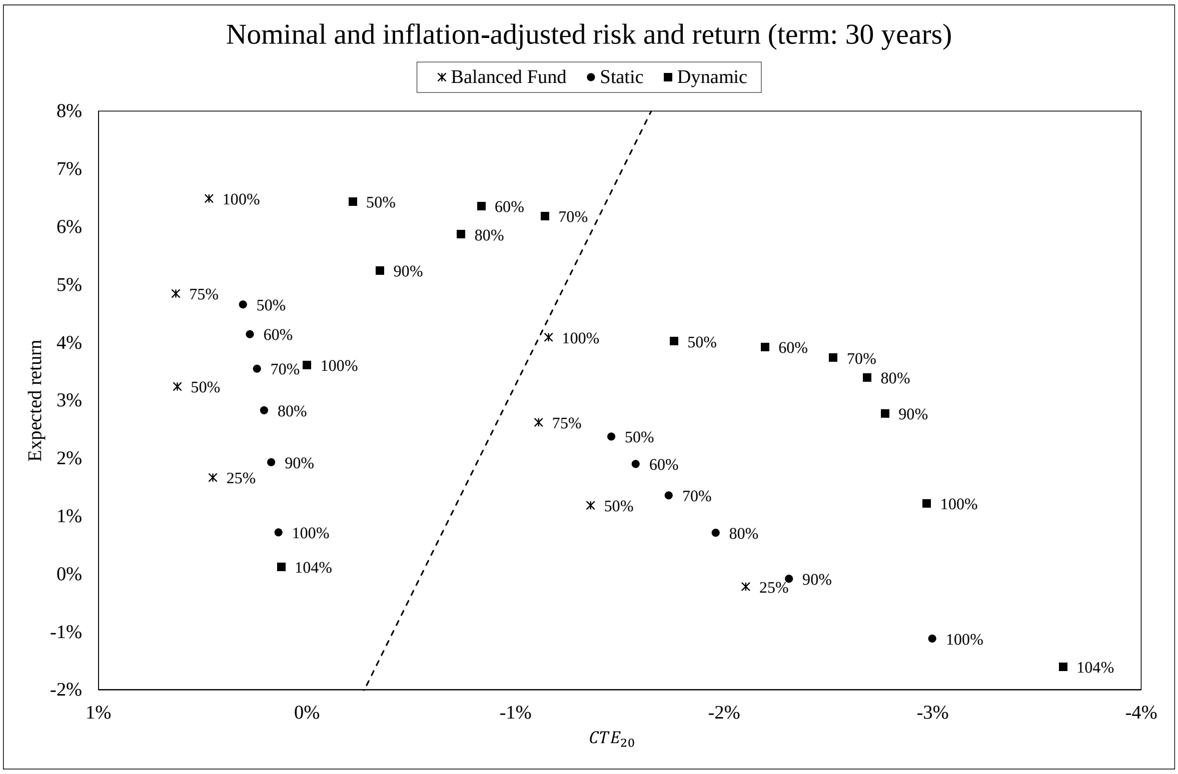

Figure 11 Nominal (upper-left part) and inflation-adjusted (lower right part) risk and return potential in case of a lower equity volatility

For a lower equity volatility and unchanged equity risk premium, the relative attractiveness of equity investments over zero bond investments increases. This can be observed throughout all result shown. Under both nominal and real terms, reducing the guarantee level of the guarantee products (or increasing the equity ratio of the balanced fund) leads to an increase in the return potential. The effect on risk apparently is highly influenced by the implicit change in the sharpe ratio. Even under nominal terms, reducing the guarantee level of the guarantee products (or increasing the equity ratio of the balanced fund) hardly shows any impact on the product's risk. Since also the importance of inflation risk (relative to equity risk) increases, the effect on real risk is even more pronounced than above: Under real terms, the risk even decreases for products with higher equity ratios.

5.2.4.3 Impact of the considered risk measure

While previous results can be explained by the mentioned effects, we should also keep in mind that we consider a very specific risk measure CTE_{20} in the analyses shown so far. Therefore, we analyze the impact of different risk measures when we move further in the tail of the considered distributions. So far, we have been focusing on the CTE_{20} which e.g. in Germany plays an important role when it comes to the risk return classification of government subsidized products (c.f. Graf and Korn, 2020). Now, we also analyze the conditional tail expectation to a level of 15%, 10% and 5%.

Figure 12 shows the nominal and inflation-adjusted ('real') risk and return potential of the considered products for the risk measures CTE_{20} (upper-left), CTE_{15} (upper-right), CTE_{10} (lower-left), and CTE_5 (lower-right) assuming the base case parameter set (cf. Table 1).

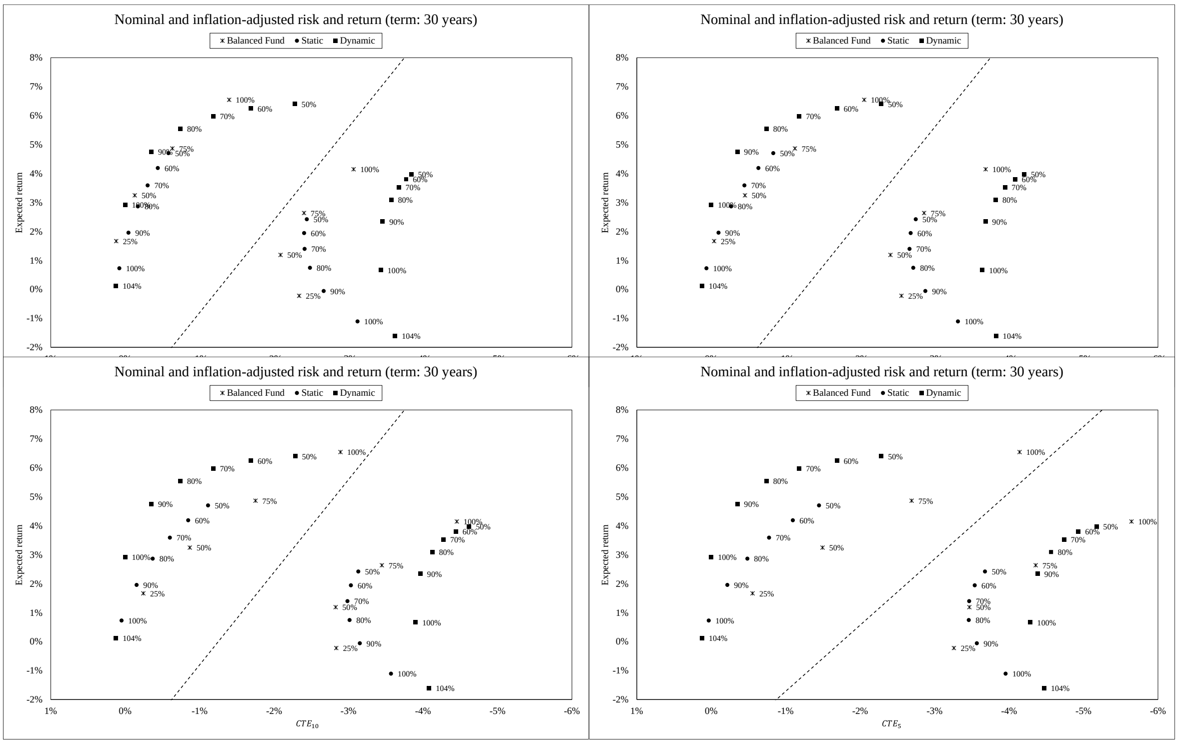

Figure 12 Nominal and inflation-adjusted ('real') risk and return potential of the considered products for the risk measures CTE_{20} (upper-left), CTE_{15} (upper-right), CTE_{10} (lower-left), and CTE_5 (lower-right) assuming the base case parameter set

For lower levels of the conditional tail expectation, obviously, the risk of all products increases: We observe a lower CTE-value if we move further into the lower tail of the distribution. The risk reducing benefit of guarantees becomes more relevant if lower levels of the conditional tail expectation are considered. First, we observe that the 'critical guarantee level' above which a further increase of the guarantee does not reduce real risk any more increases the further we move in the tail. It is worth noting, however, that even for the 5% CTE risk measure, this level is between 70% and 80% for the static and about 100% (i.e. still roughly 4 percentage points below the maximum guarantee) for the dynamic product. This coincides with an average equity ratio over the term of the product of roughly 25% which is also roughly the (real) risk minimizing equity ratio for the balanced product.

While under the base case risk measure, reducing the guarantee level of the guarantee products (or increasing the equity ratio of the balanced fund) shows only a slight increase of the product's risk under nominal terms, the nominal risk of lower guarantee levels exhibits a stronger increase if a CTE-level of 5% is considered.

Under real terms, where reducing the guarantee level of the guarantee products below the critical level (or increasing the equity ratio of the balanced fund above the real-risk minimizing level) has hardly any impact under the base case risk measure, for a lower CTE-level, the risk increases much more strongly. The events in the tail of the probability distribution are hence stronger influenced by equity risk that by inflation risk.

6 Conclusion

In this paper, we have argued that for long-term savings processes, real risk-return characteristics are more relevant to consumers than their nominal counterparts. We have also argued that – due to a positive correlation between long-term inflation and equity returns, real risk-return characteristics can be structurally different from nominal risk-return characteristics. We have analyzed a variety of typical retirement-savings products and confirmed that for popular product designs real risk indeed structurally differs from nominal risk. Hence, typically used nominal risk-return characteristics can be misleading for consumers.

While concrete results of course depend on model assumptions, model parameters, considered risk measures and product design, the effects we describe appear particularly relevant for products with rather low nominal risk. Such products are often dominated by other products in real terms, meaning that other products come with a higher real return and at the same time a lower real risk. As a rule of thumb, (very) risk averse consumers, who are looking for products with a risk close to the lowest possible risk should not buy a product with a very low nominal risk / a very high nominal guarantee, but rather go for a higher stock ratio / a lower guarantee level although such products come with higher nominal risk.

These results are relevant for product providers, financial advisors, as well as regulators: Product providers should not market the products with the lowest nominal risk (typically products with very high guarantees) as the safest options in long-term savings. Financial advisors should encourage their (risk averse) clients to invest in products with higher stock ratios (lower guarantees). Regulators should not require the use of risk-return classes that are based on nominal risk-return characteristics only.

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A Details of the financial model

In this appendix we summarize some details of the stochastic model introduced in Section 3 and especially show how to set up the model such that model-induced prices of zero-coupon bonds match those of an initial term structure of interest rates. By doing so, we especially derive the arbitrage free prices of zero-coupon bonds $P(t,T)$ in this model (cf. Appendix A.I.). In addition, we compute the correlation coefficient of equity returns and cumulated inflation which was depicted in Section 3.2.1 in Appendix A.II.

A.I Pricing of zero-coupon bonds and fitting the initial term structure of interest rates

For deriving the arbitrage-free prices $P(t,T)$ of a zero-coupon bond at time t with maturity T we first consider the underlying stochastic processes under the risk-neutral pricing measure $\mathbb{Q}$ which is obtained from the real-world measure $\mathbb{P}$ by just skipping the risk premia θ_x, θ_y and λ_S.[25] Hence, we consider

$$r(t) = x(t) + y(t) + i(t) + \psi(t)$$

with

$$dx(t) = -a_x x(t)dt + \sigma_x dW_x(t), x(0) = 0$$

$$dy(t) = -a_y y(t)dt + \sigma_y dW_y(t), y(0) = 0$$

$$di(t) = a_i\big(\theta_i - i(t)\big)dt + \sigma_i dW_i(t), i(0) = i_0$$

where $W_x(t), W_y(t)$ and $W_i(t)$ are $\mathbb{Q}$ −Brownian motions. The risk-neutral arbitrage free price of a zero-coupon bond at time t with maturity T is defined as

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T (x(s)+y(s)+i(s)+\psi(s))ds}\middle|\mathcal{F}_t\right].$$

Set $Z(t,T) \coloneqq \int_t^T \big(x(s) + y(s) + i(s)\big)ds$ and conclude that $Z(t,T)$ follows a normal distribution with some expectation and variance. Hence, to derive $P(t,T)$ for given (deterministic) $\psi(t)$ it is sufficient to analyze $Z(t,T)$ in more detail.

We obtain

$$\mathbb{E}_{\mathbb{Q}}[Z(t,T)|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}\left[\int_t^T x(s)ds\middle|\mathcal{F}_t\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_t^T y(s)ds\middle|\mathcal{F}_t\right] + \mathbb{E}_{\mathbb{Q}}\left[\int_t^T i(s)ds\middle|\mathcal{F}_t\right]$$

[25] Note, we do not incorporate any risk premium in the process for the modeling of inflation.

$$= \frac{1-e^{-a_x(T-t)}}{a_x}x(t) + \frac{1-e^{-a_y(T-t)}}{a_y}y(t) + \theta_i(T-t)$$
$$+ \frac{1-e^{-a_i(T-t)}}{a_i}(i(t)-\theta_i)$$

and

$$Var_{\mathbb{Q}}\left[Z(t,T)|\mathcal{F}_t\right] = Var_{\mathbb{Q}}\left[\int_t^T \big(x(s)+y(s)\big)ds \middle| \mathcal{F}_t\right] + Var_{\mathbb{Q}}\left[\int_t^T i(s)ds \middle| \mathcal{F}_t\right]$$
$$= \frac{\sigma_x^2}{a_x^2}\left((T-t) + \frac{2}{a_x}\mathrm{e}^{-a_x(T-t)} - \frac{1}{2a_x}\mathrm{e}^{-2a_x(T-t)} - \frac{3}{2a_x}\right)$$
$$+ \frac{\sigma_y^2}{a_y^2}\left((T-t) + \frac{2}{a_y}\mathrm{e}^{-a_y(T-t)} - \frac{1}{2a_y}\mathrm{e}^{-2a_y(T-t)} - \frac{3}{2a_y}\right)$$
$$+ 2\rho_{x,y}\frac{\sigma_x\sigma_y}{a_x a_y}\left((T-t) + \frac{1}{a_x}\left(\mathrm{e}^{-a_x(T-t)}-1\right) + \frac{1}{a_y}\left(\mathrm{e}^{-a_y(T-t)}-1\right)\right.$$
$$\left. - \frac{1}{a_x+a_Y}\left(\mathrm{e}^{-(a_x+a_y)(T-t)}-1\right)\right)$$
$$+ \frac{\sigma_i^2}{a_i^2}\left((T-t) + \frac{2}{a_i}\mathrm{e}^{-a_i(T-t)} - \frac{1}{2a_i}\mathrm{e}^{-2a_i(T-t)} - \frac{3}{2a_i}\right).$$

For ease of notation setting $V(t,T) \coloneqq Var_{\mathbb{Q}}\left[Z(t,T)|\mathcal{F}_t\right]$, we obtain

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right] = e^{-\int_t^T \psi(s)ds} \cdot e^{-\mathbb{E}_{\mathbb{Q}}[Z(t,T)|\mathcal{F}_t] + \frac{1}{2}V(t,T)}.$$

Fitting the initial term structure of interest rates

Next consider with $P^M(0,T), \forall T$ the initial term structure of interest rates as given by the market. We now want to set $\psi(T), \forall T$ such that above pricing formula coincides with the pre-specified term structure of interest rates at time 0.

Hence, we require

$$P^M(0,T) \overset{!}{=} e^{-\int_0^T \psi(s)ds} \cdot e^{-\mathbb{E}_{\mathbb{Q}}[Z(0,T)|\mathcal{F}_0] + \frac{1}{2}V(0,T)}, \forall T$$

and thus get

$$\log P^M(0,T) = -\int_0^T \psi(s)ds - \mathbb{E}_{\mathbb{Q}}[Z(0,T)|\mathcal{F}_0] + \frac{1}{2}V(0,T)$$

$$= -\int_0^T \psi(s)ds - \left(\frac{1-e^{-a_xT}}{a_x}x(0) + \frac{1-e^{-a_yT}}{a_y}y(0) + \theta_i T + \frac{1-e^{-a_iT}}{a_i}(i(0)-\theta_i)\right) + \frac{1}{2}V(0,T)$$

$$= -\int_0^T \psi(s)ds - \frac{1-e^{-a_iT}}{a_i}(i(0)-\theta_i) - \theta_i T + \frac{1}{2}V(0,T).$$

Taking derivatives with respect to time on both sides further yields

$$\frac{\partial}{\partial T}\log P^M(0,T) = -\psi(T) - e^{-a_iT}(i(0)-\theta_i) - \theta_i + \frac{\partial}{\partial T}\frac{1}{2}V(0,T)$$

$$\Leftrightarrow$$

$$f^M(0,T) = -\psi(T) - e^{-a_iT}(i(0)-\theta_i) - \theta_i + \frac{1}{2}\left(\frac{\sigma_x^2}{a_x^2}(1-e^{-a_xT})^2 + \frac{\sigma_y^2}{a_y^2}(1-e^{-a_yT})^2 + 2\rho_{x,y}\frac{\sigma_x\sigma_y}{a_xb_x}(1-e^{-a_xT})(1-e^{-b_xT}) + \frac{\sigma_i^2}{a_i^2}(1-e^{-a_iT})^2\right)$$

where $f^M(0,T) = \frac{\partial}{\partial T}\log P^M(0,T)$ denotes the instantaneous forward rate implied by the initial term structure of interest rates. Thus, we finally obtain

$$\psi(T) = f^M(0,T) + \frac{\sigma_x^2}{2a_x^2}(1-e^{-a_xT})^2 + \frac{\sigma_y^2}{2a_y^2}(1-e^{-a_yT})^2 + \rho_{x,y}\frac{\sigma_x\sigma_y}{a_xb_x}(1-e^{-a_xT})(1-e^{-a_yT}) + \frac{\sigma_i^2}{2a_i^2}(1-e^{-a_iT})^2 - e^{-a_iT}(i(0)-\theta_i) - \theta_i.$$

Pricing of zero-coupon bonds

After having set $\psi(s)$ accordingly, we are now in the position to finally derive $P(t,T)$ by

$$P(t,T) = e^{-\int_t^T \psi(s)ds}e^{-\mathbb{E}_{\mathbb{Q}}[Z(t,T)|\mathcal{F}_t] + \frac{1}{2}V(t,T)}$$

with

$$e^{-\int_t^T \psi(s)ds} = e^{-\int_0^T \psi(s)ds} \cdot e^{\int_0^t \psi(s)ds} = \frac{P^M(0,T)e^{\mathbb{E}_{\mathbb{Q}}[Z(0,T)|\mathcal{F}_0]-\frac{1}{2}V(0,T)}}{P^M(0,t)e^{\mathbb{E}_{\mathbb{Q}}[Z(0,t)|\mathcal{F}_0]-\frac{1}{2}V(0,t)}}$$

and hence obtain

$$
\begin{aligned}
P(t,T) &= \frac{P^M(0,T)e^{\mathbb{E}_{\mathbb{Q}}[Z(0,T)|\mathcal{F}_0]-\frac{1}{2}V(0,T)}}{P^M(0,t)e^{\mathbb{E}_{\mathbb{Q}}[Z(0,t)|\mathcal{F}_0]-\frac{1}{2}V(0,t)}} e^{-\mathbb{E}_{\mathbb{Q}}[Z(t,T)|\mathcal{F}_t]+\frac{1}{2}V(t,T)} \\
&= \frac{P^M(0,T)}{P^M(0,t)} \exp\big(A(t,T)\big)
\end{aligned}
$$

with

$$
\begin{aligned}
A(t,T) = {} & \frac{1}{2}\big(V(t,T)-V(0,T)+V(0,t)\big) - \frac{1-e^{-a_x(T-t)}}{a_x}x(t) - \frac{1-\mathrm{e}^{-a_y(T-t)}}{a_y}y(t) \\
& - \theta_i(T-t) - \frac{1-e^{-a_i(T-t)}}{a_i}\big(i(t)-\theta_i\big) + \theta_i T + \frac{1-e^{-a_i T}}{a_i}\big(i(0)-\theta_i\big) \\
& - \theta_i t - \frac{1-e^{-a_i t}}{a_i}\big(i(0)-\theta_i\big) \\
= {} & \frac{1}{2}\big(V(t,T)-V(0,T)+V(0,t)\big) - \frac{1-e^{-a_x(T-t)}}{a_x}x(t) - \frac{1-e^{-a_y(T-t)}}{a_y}y(t) \\
& - \frac{1-e^{-a_i(T-t)}}{a_i}\big(i(t)-\theta_i\big) + \big(i(0)-\theta_i\big)\left(\frac{1-e^{-a_i(T)}}{a_i} - \frac{1-e^{-a_i(t)}}{a_i}\right).
\end{aligned}
$$

A.II Correlation of cumulated equity returns and cumulated inflation

In this section we derive the correlation of $\ln S_A(T)$ and $\ln CPI(T)$ in our model , i.e. we derive

$$
Corr\big(\ln(S_A(T)), \ln(CPI(T))\big) := \frac{Cov\big(\ln(S_A(T)), \ln(CPI(T))\big)}{\sqrt{Var(\ln S_A(T)) \cdot Var\,\big(\ln(CPI(T))\big)}}.
$$

For doing so, we study the covariance of $\ln\big(S_A(T)\big)$ and $\ln\big(CPI(T)\big)$ and obtain

$$
\begin{aligned}
& Cov\big(\ln(S_A(T)), \ln(CPI(T))\big) \\
& = Cov\left(\int_0^T r(s)ds + (\lambda_A - 0.5\sigma_A^2)T + \sigma_A W_S(T), \int_0^T i(s)ds\right) \\
& = Cov\left(\int_0^T \big(x(s)+y(s)+i(s)+\psi(s)\big)ds + \sigma_A W_S(T), \int_0^T i(s)ds\right) \\
& = Cov\left(\int_0^T \big(x(s)+y(s)+i(s)\big)ds + \sigma_A W_S(T), \int_0^T i(s)ds\right).
\end{aligned}
$$

If we assume that $dW_x dW_i = dW_y dW_i = 0$ and $dW_S dW_i = \rho_{Si}$ hold, we further get

$$Cov\big(\ln\big(S_A(T)\big),\ln\big(CPI(T)\big)\big) = Cov\left(\int_0^T i(s)ds + \sigma_A W_S(T), \int_0^T i(s)ds\right)$$
$$= Cov\left(\int_0^T \sigma_A dW_S(u)\,, \int_0^T i(s)ds\right) + Var\left(\int_0^T i(s)ds\right).$$

and obtain

$$\begin{aligned}
&Cov\left(\int_0^T \sigma_A dW_S(u)\,, \int_0^T i(s)ds\right)\\
&= Cov\left(\int_0^T \sigma_A dW_S(u)\,, \int_0^T i(s)ds\right)\\
&= Cov\left(\int_0^T \sigma_A dW_S(u)\,, \int_0^T \left(e^{-a_i s} i(0) + \theta_i(1-e^{-a_i s}) + \int_0^s \sigma_i e^{-a_i(s-u)} dW_i(u)\right) ds\right)\\
&= Cov\left(\int_0^T \sigma_A dW_S(u)\,, \int_0^T \left(\int_0^s \sigma_i e^{-a_i(s-u)} dW_i(u)\right) ds\right)\\
&= \sigma_A \sigma_i \int_0^T Cov\left(\int_0^T dW_S(u)\,, \left(\int_0^s e^{-a_i(s-u)} dW_i(u)\right)\right) ds\\
&= \sigma_A \sigma_i \int_0^T Cov\left(\int_0^s dW_S(u)\,, \left(\int_0^s e^{-a_i(s-u)} dW_i(u)\right)\right) ds\\
&= \sigma_A \sigma_i \int_0^T \left(\int_0^s \rho_{Si} \cdot e^{-a_i(s-u)} du\right) ds = \sigma_A \sigma_i \rho_{Si} \int_0^T \frac{1}{a_i}(1-e^{-a_i s}) ds\\
&= \frac{\sigma_A \sigma_i \rho_{Si}}{a_i}\left(T + \frac{1}{a_i}(e^{-a_i T} - 1)\right)
\end{aligned}$$

Finally, the variances $Var(\ln S_A(T))$ and $Var(\ln CPI(T))$ are derived as

$$\begin{aligned}
&Var(\ln S_A(T))\\
&= Var\left(\int_0^T r(s)ds + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T + \sigma_A W_S(T)\right)\\
&= Var\left(\int_0^T x(s) + y(s) + i(s)ds + \sigma_A W_S(T)\right)\\
&= Var\left(\int_0^T x(s) + y(s)ds\right) + Var\left(\int_0^T i(s)ds + \sigma_A W_S(T)\right)\\
&= Var\left(\int_0^T x(s) + y(s)ds\right) + Var\left(\int_0^T i(s)ds\right) + 2\mathrm{Cov}\left(\int_0^T i(s)ds\,, \sigma_A W_S(T)\right) + \sigma_A^2 T
\end{aligned}$$

$$= \frac{\sigma_x^2}{a_x^2}\left(T + \frac{2}{a_x}e^{-a_xT} - \frac{1}{2a_x}e^{-2a_xT} - \frac{3}{2a_x}\right) + \frac{\sigma_y^2}{a_y^2}\left(T + \frac{2}{a_y}e^{-a_yT} - \frac{1}{2a_y}e^{-2a_yT} - \frac{3}{2a_y}\right)$$
$$+ \frac{2\rho_{x.y}\sigma_x\sigma_y}{a_xa_y}\left(T + \frac{(e^{-a_xT}-1)}{a_x} + \frac{(e^{-a_yT}-1)}{a_y} - \frac{\left(e^{-(a_x+a_y)T}-1\right)}{a_x+a_y}\right)$$
$$+ \frac{\sigma_i^2}{a_i^2}\left(T + \frac{2}{a_i}e^{-a_iT} - \frac{1}{2a_i}e^{-2a_iT} - \frac{3}{2a_i}\right)$$
$$+ 2\frac{\sigma_A\sigma_i\rho_{Si}}{a_i}\left(T + \frac{1}{a_i}(e^{-a_iT}-1)\right) + \sigma_A^2T$$

and

$$Var(\ln CPI(T)) = Var\left(\int_0^T i(s)ds\right) = \frac{\sigma_i^2}{a_i^2}\left(T + \frac{2}{a_i}\mathrm{e}^{-a_iT} - \frac{1}{2a_i}\mathrm{e}^{-2a_iT} - \frac{3}{2a_i}\right).$$

B Solution to the Merton-Problem

In this appendix we consider the famous Merton investment problem (cf. Merton, 1969) in the financial model as described in Section 3. Hence, we consider a strategy which continuously rebalances the account value $A_\alpha(t)$ between the safe asset (bank account) $C(t)$ and the equity investment assuming a constant equity ratio of $\alpha \in [0,1]$ and a volatility of σ_A and a risk premium of λ_A accordingly. The account value of this strategy hence obeys the following dynamics:

$$dA_\alpha(t) = A_\alpha(t)\big((r(t) + \alpha\lambda_A)dt + \alpha\sigma_A dW_S\big), A_\alpha(0) = 1.$$

In the financial model introduced in Section 3 the account value $A_\alpha(T)$ after T years can be solved as

$$A_\alpha(T) = \exp\left(\int_0^T r(s)ds + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T + \alpha\sigma_A W_S(T)\right).$$

Accordingly, we obtain the inflation-adjusted value $\tilde{A}_\alpha(T) = \frac{A_\alpha(T)}{CPI(T)}$ of this process as

$$\tilde{A}_\alpha(T) = \exp\left(\int_0^T r(s)ds - \int_0^T i(s)ds + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T + \alpha\sigma_A W_S(T)\right)$$
$$= \exp\left(\int_0^T \bar{r}(s)ds + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T + \alpha\sigma_A W_S(T)\right),$$

where we set $\bar{r}(s) := x(s) + y(s) + \psi(s)$.

Note, $A_\alpha(T)$ and $\tilde{A}_\alpha(T)$ both follow a log-normal probability distribution. Hence, for some normal distributed random variables $\mathcal{N}_{A_\alpha(T)}$ and $\mathcal{N}_{\tilde{A}_\alpha(T)}$ we have $A_\alpha(T) \overset{d}{=} e^{\mathcal{N}_{A_\alpha(T)}}$ and $\tilde{A}_\alpha(T) \overset{d}{=} e^{\mathcal{N}_{\tilde{A}_\alpha(T)}}$. The expectation and variance of these normal distributed random variables are derived as

$$\mathbb{E}\left[\mathcal{N}_{A_\alpha(T)}\right] = \mathbb{E}\left[\int_0^T r(s)ds\right] + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T\,,$$
$$Var\left(\mathcal{N}_{A_\alpha(T)}\right) = Var\left(\int_0^T r(s)ds\right) + \alpha^2\sigma_A^2 T$$

and

$$\mathbb{E}\left[\mathcal{N}_{\tilde{A}_\alpha(T)}\right] = \mathbb{E}\left[\int_0^T \bar{r}(s)ds\right] + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T,$$
$$Var\left(\mathcal{N}_{\tilde{A}_\alpha(T)}\right) = Var\left(\int_0^T \bar{r}(s)ds\right) + \alpha^2\sigma_A^2 T.$$

If we consider $\gamma \neq 1$,[26] the expected utility derived from A_T is

$$\mathbb{E}\left[u_\gamma\left(A_\alpha(T)\right)\right] = \mathbb{E}\left[\frac{A_\alpha(T)^{1-y}}{1-\gamma}\right] = \frac{1}{1-\gamma}\mathbb{E}\left[e^{(1-\gamma)\mathcal{N}_{A_\alpha(T)}}\right]$$
$$= \frac{1}{1-\gamma} e^{(1-\gamma)\mathbb{E}\left[\mathcal{N}_{A_\alpha(T)}\right] + 0.5(1-\gamma)^2 Var\left(\mathcal{N}_{A_\alpha(T)}\right)}.$$

Note, for fixed γ the expectation and variance of $\mathcal{N}_{A_\alpha(T)}$ are a function of the equity ratio α and hence the above expected utility takes its maximum for the maximum of

$$(1-\gamma)\mathbb{E}\left[\mathcal{N}_{A_\alpha(T)}\right] + 0.5(1-\gamma)^2 Var\left(\mathcal{N}_{A_\alpha(T)}\right)$$
$$= (1-\gamma)\left(\mathbb{E}\left[\int_0^T r(s)ds\right] + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T\right)$$
$$+ 0.5(1-\gamma)^2\left(Var\left(\int_0^T r(s)ds\right) + \alpha^2\sigma_A^2 T\right)$$

which takes its maximum for the same α as the function

$$(1-\gamma)\left((\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T\right) + 0.5(1-\gamma)^2(\alpha^2\sigma_A^2 T).$$

Setting the first derivative with respect to α to zero yields

$$0 \overset{!}{=} (1-\gamma)(\lambda_A - \sigma_A^2\alpha)T + (1-\gamma)^2\sigma_A^2 T\alpha$$

[26] The case for $\gamma = 1$ naturally follows the same derivations and is hence omitted here.

$$\Leftrightarrow$$

$$\alpha = \frac{-(1-\gamma)\lambda_A T}{(1-\gamma)(-\sigma_A^2 T + (1-\gamma)\sigma_A^2 T)} = \frac{\lambda_A}{\gamma\sigma_A^2}$$

which equals the well-known solution of the Merton-problem.

Regarding the expected utility of $\tilde{A}_\alpha(T)$ for fixed γ we arrive at maximizing

$$\begin{aligned}(1-\gamma)\mathbb{E}\left[\mathcal{N}_{\tilde{A}_\alpha(T)}\right] &+ 0.5(1-\gamma)^2 Var\left(\mathcal{N}_{\tilde{A}_\alpha(T)}\right)\\ &= (1-\gamma)\left(\mathbb{E}\left[\int_0^T \bar{r}(s)ds\right] + (\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T\right)\\ &+ 0.5(1-\gamma)^2\left(Var\left(\int_0^T \bar{r}(s)ds\right) + \alpha^2\sigma_A^2 T\right)\end{aligned}$$

which takes its maximum for the same α as the function

$$(1-\gamma)\big((\alpha\lambda_A - 0.5\alpha^2\sigma_A^2)T\big) + 0.5(1-\gamma)^2(\alpha^2\sigma_A^2 T)$$

and hence at the same equity ratio

$$\alpha = \frac{\lambda_A}{\gamma\sigma_A^2}.$$

Thus, in our considered modelling framework the optimal equity ratio in the Merton problem is the same for the nominal and inflation-adjusted view. However, bear in mind that this is a consequence of the model limitations stated in Section 3.2 which imply that the pure money market account $C(t)$ similar to a 'real' equity investment serves as an inflation hedge in our model and hence the only relevant figure which enters the Merton optimization in the inflation-adjusted view is the volatility of the underlying assets. Therefore, it should be of no surprise that the results don't change when we change our perspective from a nominal to an inflation-adjusted view.

In summary, these results should however due to the modelling effects and the implicit assumptions on the pure money market be treated with the necessary caution.